



# Real hypersurfaces in the complex quadric with Reeb invariant Ricci tensor



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## ABSTRACT

We introduce the notion of Reeb invariant Ricci tensor for real hypersurfaces in the complex quadric  $Q^m = SO_{m+2}/SO_mSO_2$ . The Reeb invariant Ricci tensor implies that the unit normal vector field  $N$  becomes  $\mathfrak{A}$ -principal or  $\mathfrak{A}$ -isotropic. Then according to each case, we give a complete classification of real hypersurfaces in  $Q^m = SO_{m+2}/SO_mSO_2$  with Reeb invariant Ricci tensor.

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## 1. Introduction

When we consider some Hermitian symmetric spaces of rank 2, we can usually give examples of Riemannian symmetric spaces  $SU_{m+2}/S(U_2U_m)$  and  $SU_{2,m}/S(U_2U_m)$ , which are said to be complex two-plane Grassmannians and complex hyperbolic two-plane Grassmannians respectively (see [1–4] and [5]). These are viewed as Hermitian symmetric spaces and quaternionic Kähler symmetric spaces equipped with the Kähler structure  $J$  and the quaternionic Kähler structure  $\mathfrak{J}$  and they have rank 2.

As another kind of Hermitian symmetric space with rank 2 of compact type different from the above ones, we can give an example of complex quadric  $Q^m = SO_{m+2}/SO_mSO_2$ , which is a complex hypersurface in complex projective space  $\mathbb{C}P^{m+1}$  (see Klein [6], and Smyth [7]). The complex quadric can also be regarded as a kind of real Grassmann manifolds of compact type with rank 2 (see Kobayashi and Nomizu [8]). Accordingly, the complex quadric admits two important geometric structures, a complex conjugation structure  $A$  and a Kähler structure  $J$ , which anti-commute with each other, that is,  $AJ = -JA$ . Then for  $m \geq 2$  the triple  $(Q^m, J, g)$  is a Hermitian symmetric space of compact type with rank 2 and its maximal sectional curvature is equal to 4 (see Klein [6] and Reckziegel [9]).

In the complex projective space  $\mathbb{C}P^{m+1}$  and the quaternionic projective space  $\mathbb{Q}P^{m+1}$  some classifications related to commuting Ricci tensor were investigated by Kimura [10,11], Pérez [12] and Pérez and Suh [13,14] respectively. The classification problems of the complex 2-plane Grassmannian  $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_2U_m)$  with certain geometric

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conditions were mainly discussed in Suh [3,15] and [4], where the classification of *contact hypersurfaces*, *parallel Ricci tensor* and *harmonic curvature* of a real hypersurface in  $G_2(\mathbb{C}^{m+2})$  were extensively studied. Moreover, in [4] we have asserted that the Reeb flow on a real hypersurface in  $SU_{2,m}/S(U_2U_m)$  is isometric if and only if  $M$  is an open part of a tube around a totally geodesic  $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$ . Suh [5] strengthened this result to hypersurfaces in  $G_2(\mathbb{C}^{m+2})$  with commuting Ricci tensor and gave a characterization of real hypersurfaces in  $G_2(\mathbb{C}^{m+2}) = SU_{m+2}/S(U_mU_2)$  as follows:

**Theorem A.** *Let  $M$  be a Hopf real hypersurface in  $G_2(\mathbb{C}^{m+2})$  with commuting Ricci tensor,  $m \geq 3$ . Then  $M$  is locally congruent to a tube of radius  $r$  over a totally geodesic  $G_2(\mathbb{C}^{m+1})$  in  $G_2(\mathbb{C}^{m+2})$ .*

On the other hand, Suh and Woo [16] have investigated a classification problem of real hypersurfaces in  $SU_{2,m}/S(U_2 \cdot U_m)$  with parallel Ricci tensor. Moreover, Suh [5] studied another classification for Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians  $SU_{2,m}/S(U_2U_m)$  with commuting Ricci tensor as follows:

**Theorem B.** *Let  $M$  be a Hopf hypersurface in  $SU_{2,m}/S(U_2 \cdot U_m)$  with commuting Ricci tensor,  $m \geq 3$ . Then  $M$  is locally congruent to an open part of a tube around some totally geodesic  $SU_{2,m-1}/S(U_2 \cdot U_{m-1})$  in  $SU_{2,m}/S(U_2 \cdot U_m)$  or a horosphere whose center at infinity with  $JX \in \mathfrak{J}X$  is singular.*

It is known that the Reeb flow on a real hypersurface in  $G_2(\mathbb{C}^{m+2})$  is isometric if and only if  $M$  is an open part of a tube around a totally geodesic  $G_2(\mathbb{C}^{m+1}) \subset G_2(\mathbb{C}^{m+2})$  in [5] and [17]. Moreover, in [4] we asserted that the Reeb flow on a real hypersurface in  $SU_{2,m}/S(U_2U_m)$  is isometric if and only if  $M$  is an open part of a tube around a totally geodesic  $SU_{2,m-1}/S(U_2U_{m-1}) \subset SU_{2,m}/S(U_2U_m)$ . Here, the Reeb flow on real hypersurfaces in  $SU_{m+2}/S(U_mU_2)$  or  $SU_{2,m}/S(U_2U_m)$  is said to be *isometric* if the shape operator commutes with the structure tensor. Then naturally it can be easily checked that the Ricci tensor commutes with the structure tensor. In the paper [18] due to Suh and Hwang, we investigated this problem for real hypersurfaces in the complex quadric  $Q^m = SO_{m+2}/SO_mSO_2$  and obtained the following result:

**Theorem C.** *Let  $M$  be a Hopf real hypersurface in the complex quadric  $Q^m$ ,  $m \geq 4$ , with commuting Ricci tensor. If the shape operator commutes with the structure tensor on the distribution  $\mathcal{Q}^\perp$ , then  $M$  is locally congruent to an open part of a tube around totally geodesic  $\mathbb{CP}^k$  in  $Q^{2k}$ ,  $m = 2k$  or  $M$  has 3 distinct constant principal curvatures given by*

$$\alpha = \sqrt{2(m-3)}, \gamma = 0, \lambda = 0, \text{ and } \mu = -\frac{2}{\sqrt{2(m-3)}} \text{ or}$$

$$\alpha = \sqrt{\frac{2}{3}}(m-3), \gamma = 0, \lambda = 0, \text{ and } \mu = -\frac{\sqrt{6}}{\sqrt{m-3}}$$

with corresponding principal curvature spaces respectively

$$T_\alpha = [\xi], T_\gamma = [A\xi, AN], \phi(T_\lambda) = T_\mu, \text{ and } \dim T_\lambda = \dim T_\mu = m-2.$$

Now at each point  $z \in M$  let us consider a maximal  $\mathfrak{A}$ -invariant subspace  $\mathcal{Q}_z$  of  $T_zM$ ,  $z \in M$ , defined by

$$\mathcal{Q}_z = \{X \in T_zM \mid AX \in T_zM \text{ for all } A \in \mathfrak{A}_z\}$$

of  $T_zM$ ,  $z \in M$ . Thus for a case where the unit normal vector field  $N$  is  $\mathfrak{A}$ -isotropic it can be easily checked that the orthogonal complement  $\mathcal{Q}_z^\perp = \mathcal{C}_z \ominus \mathcal{Q}_z$ ,  $z \in M$ , of the distribution  $\mathcal{Q}$  in the complex subbundle  $\mathcal{C}$ , becomes  $\mathcal{Q}_z^\perp = \text{Span}\{A\xi, AN\}$ . Here it can be easily checked that the vector fields  $A\xi$  and  $AN$  belong to the tangent space  $T_zM$ ,  $z \in M$  if the unit normal vector field  $N$  becomes  $\mathfrak{A}$ -isotropic. Thus for a case where the unit normal vector field  $N$  is  $\mathfrak{A}$ -isotropic it can be easily checked that the orthogonal complement  $\mathcal{Q}_z^\perp = \mathcal{C}_z \ominus \mathcal{Q}_z$ ,  $z \in M$ , of the distribution  $\mathcal{Q}$  in the complex subbundle  $\mathcal{C}$ , becomes  $\mathcal{Q}_z^\perp = \text{Span}\{A\xi, AN\}$ . Moreover, the vector fields  $A\xi$  and  $AN$  belong to the tangent space  $T_zM$ ,  $z \in M$  if the unit normal vector field  $N$  becomes  $\mathfrak{A}$ -isotropic. Then motivated by the above result, in [17] we gave another theorem for real hypersurfaces in the complex quadric  $Q^m$  with parallel Ricci tensor and  $\mathfrak{A}$ -isotropic unit normal.

Apart from the complex structure  $J$  there is another distinguished geometric structure on  $Q^m$ , namely a parallel rank two vector bundle  $\mathfrak{A}$  which contains an  $S^1$ -bundle of real structures, that is, complex conjugations  $A$  on the tangent spaces of  $Q^m$ . This geometric structure determines a maximal  $\mathfrak{A}$ -invariant subbundle  $\mathcal{Q}$  of the tangent bundle  $TM$  of a real hypersurface  $M$  in  $Q^m$ .

Recall that a nonzero tangent vector  $W \in T_{[z]}Q^m$  is called singular if it is tangent to more than one maximal flat in  $Q^m$ . There are two types of singular tangent vectors for the complex quadric  $Q^m$ :

1. If there exists a conjugation  $A \in \mathfrak{A}$  such that  $W \in V(A)$ , then  $W$  is singular. Such a singular tangent vector is called  $\mathfrak{A}$ -principal.
2. If there exist a conjugation  $A \in \mathfrak{A}$  and orthonormal vectors  $X, Y \in V(A)$  such that  $W/\|W\| = (X + JY)/\sqrt{2}$ , then  $W$  is singular. Such a singular tangent vector is called  $\mathfrak{A}$ -isotropic.

When we consider a hypersurface  $M$  in the complex quadric  $Q^m$ , under the assumption of some geometric properties the unit normal vector field  $N$  of  $M$  in  $Q^m$  can be divided into two classes if either  $N$  is  $\mathfrak{A}$ -isotropic or  $\mathfrak{A}$ -principal (see [19] and [17]). In the first case where  $N$  is  $\mathfrak{A}$ -isotropic, we have shown in [19] that  $M$  is locally congruent to a tube over a totally

geodesic  $\mathbb{C}P^k$  in  $Q^{2k}$ . In the second case, when the unit normal  $N$  is  $\mathfrak{A}$ -principal, we proved that a contact hypersurface  $M$  in  $Q^m$  is locally congruent to a tube over a totally geodesic and totally real submanifold  $S^m$  in  $Q^m$  (see [17]).

In the study of complex two-plane Grassmannian  $G_2(\mathbb{C}^{m+2})$  or complex hyperbolic two-plane Grassmannian  $SU_{2,m}/S(U_2 \cdot U_m)$  we studied hypersurfaces with parallel Ricci tensor and gave non-existence properties respectively (see [3] and [16]). In [17] we also considered the notion of parallel Ricci tensor, that is,  $\nabla \text{Ric} = 0$ , for hypersurfaces  $M$  in  $Q^m$ . But from the assumption of Ricci parallel, it was impossible to derive the fact that either the unit normal  $N$  is  $\mathfrak{A}$ -isotropic or  $\mathfrak{A}$ -principal. So in [17] we gave a classification with the further assumption of  $\mathfrak{A}$ -isotropic.

But fortunately when we consider Ricci commuting, that is,  $\text{Ric} \cdot \phi = \phi \cdot \text{Ric}$  for hypersurfaces  $M$  in  $Q^m$ , we can assert that the unit normal vector field  $N$  becomes either  $\mathfrak{A}$ -isotropic or  $\mathfrak{A}$ -principal (see Suh and Hwang [18]). Then motivated by such a result and using Theorem C, in this paper we give a complete classification for real hypersurfaces in the complex quadric  $Q^m$  with Reeb invariant Ricci tensor, that is,  $\mathcal{L}_\xi \text{Ric} = 0$  as follows:

**Main Theorem.** *Let  $M$  be a Hopf real hypersurface in the complex quadric  $Q^m$ ,  $m \geq 4$ , with Reeb invariant Ricci tensor. If the shape operator commutes with the structure tensor on the distribution  $Q^\perp$ , then  $M$  is locally congruent to an open part of a tube around totally geodesic  $\mathbb{C}P^k$  in  $Q^{2k}$ ,  $m = 2k$  or  $M$  has 3 distinct constant principal curvatures given by*

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with corresponding principal curvature spaces respectively

$$T_\alpha = [\xi], T_\gamma = [A\xi, AN], \phi(T_\lambda) = T_\mu, \text{ and } \dim T_\lambda = \dim T_\mu = m-2.$$

**Remark 1.1.** In Main Theorem the second and third ones can be explained geometrically as follows: the real hypersurface  $M$  is locally congruent to  $M_1 \times \mathbb{C}$ , where  $M_1$  is a tube of radius  $r = \frac{1}{\sqrt{2}} \tan^{-1} \sqrt{m-3}$  or respectively, of radius  $r = \frac{1}{\sqrt{2}} \tan^{-1} \sqrt{\frac{m-3}{3}}$ , over an  $m-1$ -dimensional unit sphere  $S^{m-1}$  in  $Q^{m-1}$ . Then, by the result due to Suh [17],  $M_1$  becomes a contact hypersurface defined by  $S\phi + \phi S = k\phi$ ,  $k = -\frac{2}{\sqrt{2(m-3)}}$ , and  $k = -\frac{\sqrt{6}}{\sqrt{m-3}}$  respectively. By using the Segre embedding, the embedding  $M_1 \times \mathbb{C} \subset Q^{m-1} \times \mathbb{C} \subset Q^m$  is defined by  $(z_0, z_1, \dots, z_m, w) \rightarrow (z_0 w, z_1 w, \dots, z_m w, 0)$ . Here  $(z_0 w)^2 + (z_1 w)^2 + \dots + (z_m w)^2 = (z_0^2 + \dots + z_m^2)w^2 = 0$ , where  $\{z_0, \dots, z_m\}$  denotes a coordinate system in  $Q^{m-1}$  satisfying  $z_0^2 + \dots + z_m^2 = 0$ .

## 2. The complex quadric

For more background to this section we refer to [6,8,9,17,19] and [20]. The complex quadric  $Q^m$  is the complex hypersurface in  $\mathbb{C}P^{m+1}$  which is defined by the equation  $z_0^2 + \dots + z_{m+1}^2 = 0$ , where  $z_0, \dots, z_{m+1}$  are homogeneous coordinates on  $\mathbb{C}P^{m+1}$ . We equip  $Q^m$  with the Riemannian metric  $g$  which is induced from the Fubini–Study metric  $\bar{g}$  on  $\mathbb{C}P^{m+1}$  with constant holomorphic sectional curvature 4. The Fubini–Study metric  $\bar{g}$  is defined by  $\bar{g}(X, Y) = \Phi(JX, Y)$  for any vector fields  $X$  and  $Y$  on  $\mathbb{C}P^{m+1}$  and a globally closed  $(1, 1)$ -form  $\Phi$  given by  $\Phi = -4i\partial\bar{\partial} \log f_j$  on an open set  $U_j = \{[z^0, z^1, \dots, z^{m+1}] \in \mathbb{C}P^{m+1} | z^j \neq 0\}$ , where the function  $f_j$  denotes  $f_j = \sum_{k=0}^{m+1} t_j^k \bar{t}_j^k$ , and  $t_j^k = \frac{z^k}{z^j}$  for  $j, k = 0, \dots, m+1$ . Then naturally the Kähler structure on  $\mathbb{C}P^{m+1}$  induces canonically a Kähler structure  $(J, g)$  on the complex quadric  $Q^m$ .

The complex projective space  $\mathbb{C}P^{m+1}$  is a Hermitian symmetric space of the special unitary group  $SU_{m+2}$ , namely  $\mathbb{C}P^{m+1} = SU_{m+2}/S(U_{m+1}U_1)$ . We denote by  $o = [0, \dots, 0, 1] \in \mathbb{C}P^{m+1}$  the fixed point of the action of the stabilizer  $S(U_{m+1}U_1)$ . The special orthogonal group  $SO_{m+2} \subset SU_{m+2}$  acts on  $\mathbb{C}P^{m+1}$  with cohomogeneity one. The orbit containing  $o$  is a totally geodesic real projective space  $\mathbb{R}P^{m+1} \subset \mathbb{C}P^{m+1}$ . The second singular orbit of this action is the complex quadric  $Q^m = SO_{m+2}/SO_m SO_2$ . This homogeneous space model leads to the geometric interpretation of the complex quadric  $Q^m$  as the Grassmann manifold  $G_2^+(\mathbb{R}^{m+2})$  of oriented 2-planes in  $\mathbb{R}^{m+2}$ . It also gives a model of  $Q^m$  as a Hermitian symmetric space of rank 2. The complex quadric  $Q^1$  is isometric to a sphere  $S^2$  with constant curvature, and  $Q^2$  is isometric to the Riemannian product of two 2-spheres with constant curvature. For this reason we will assume  $m \geq 3$  from now on.

In another way, the complex projective space  $\mathbb{C}P^{m+1}$  is defined by using the Hopf fibration

$$\pi : S^{2m+3} \rightarrow \mathbb{C}P^{m+1}, \quad z \rightarrow [z],$$

which is said to be a Riemannian submersion. Then naturally we can consider the following diagram for the complex quadric  $Q^m$  as follows:

$$\begin{array}{ccc} \tilde{Q} = \pi^{-1}(Q) & \xrightarrow{\bar{i}} & S^{2m+3} \subset \mathbb{C}^{m+2} \\ \pi \downarrow & & \pi \downarrow \\ Q = Q^m & \xrightarrow{i} & \mathbb{C}P^{m+1} \end{array}$$

The submanifold  $\tilde{Q}$  of codimension 2 in  $S^{2m+3}$  is called the Stiefel manifold of orthonormal 2-frames in  $\mathbb{R}^{m+2}$ , which is given by

$$\tilde{Q} = \left\{ x + iy \in \mathbb{C}^{m+2} \mid g(x, x) = g(y, y) = \frac{1}{2} \text{ and } g(x, y) = 0 \right\},$$

where  $g(x, y) = \sum_{i=1}^{m+2} x_i y_i$  for any  $x = (x_1, \dots, x_{m+2})$  and  $y = (y_1, \dots, y_{m+2}) \in \mathbb{R}^{m+2}$ . Then the tangent space is decomposed as  $T_z S^{2m+3} = H_z \oplus F_z$  and  $T_z \tilde{Q} = H_z(Q) \oplus F_z(Q)$  at  $z = x + iy \in \tilde{Q}$  respectively, where the horizontal subspaces  $H_z$  and  $H_z(Q)$  are given by  $H_z = (\mathbb{C}z)^\perp$  and  $H_z(Q) = (\mathbb{C}z \oplus \mathbb{C}\bar{z})^\perp$ , and  $F_z$  and  $F_z(Q)$  are fibers which are isomorphic to each other. Here  $H_z(Q)$  becomes a subspace of  $H_z$  of real codimension 2 and orthogonal to the two unit normals  $-\bar{z}$  and  $-J\bar{z}$ . Explicitly, at the point  $z = x + iy \in \tilde{Q}$  it can be described as

$$H_z = \{u + iv \in \mathbb{C}^{m+2} \mid g(x, u) + g(y, v) = 0, \quad g(x, v) = g(y, u)\}$$

and

$$H_z(Q) = \{u + iv \in H_z \mid g(u, x) = g(u, y) = g(v, x) = g(v, y) = 0\},$$

where  $\mathbb{C}^{m+2} = \mathbb{R}^{m+2} \oplus i\mathbb{R}^{m+2}$ , and  $g(u, x) = \sum_{i=1}^{m+2} u_i x_i$  for any  $u = (u_1, \dots, u_{m+2})$ ,  $x = (x_1, \dots, x_{m+2}) \in \mathbb{R}^{m+2}$ .

These spaces can be naturally projected by the differential map  $\pi_*$  as  $\pi_* H_z = T_{\pi(z)} \mathbb{C}P^{m+1}$  and  $\pi_* H_z(Q) = T_{\pi(z)} Q$  respectively. This gives that at the point  $\pi(z) = [z]$  the tangent subspace  $T_{[z]} Q^m$  becomes a complex subspace of  $T_{[z]} \mathbb{C}P^{m+1}$  with complex codimension 1 and has two unit normal vector fields  $-\bar{z}$  and  $-J\bar{z}$  (see Reckziegel [9]).

Then let us denote by  $A_{\bar{z}}$  the shape operator of  $Q^m$  in  $\mathbb{C}P^{m+1}$  with respect to the unit normal  $\bar{z}$ . It is defined by  $A_{\bar{z}} w = \bar{\nabla}_w \bar{z} = \bar{w}$  for a complex Euclidean connection  $\bar{\nabla}$  induced from  $\mathbb{C}^{m+2}$  and all  $w \in T_{[z]} Q^m$ . That is, the shape operator  $A_{\bar{z}}$  is just a complex conjugation restricted to  $T_{[z]} Q^m$ . Moreover, it satisfies the following for any  $w \in T_{[z]} Q^m$  and any  $\lambda \in S^1 \subset \mathbb{C}$

$$\begin{aligned} A_{\lambda\bar{z}}^2 w &= A_{\lambda\bar{z}} A_{\lambda\bar{z}} w = A_{\lambda\bar{z}} \lambda \bar{w} \\ &= \lambda A_{\bar{z}} \lambda \bar{w} = \lambda \bar{\nabla}_{\lambda\bar{w}} \bar{z} = \lambda \bar{\lambda} \bar{w} \\ &= |\lambda|^2 w = w. \end{aligned}$$

Accordingly,  $A_{\lambda\bar{z}}^2 = I$  for any  $\lambda \in S^1$ . So the shape operator  $A_{\bar{z}}$  becomes an anti-commuting involution such that  $A_{\bar{z}}^2 = I$  and  $AJ = -JA$  on the complex vector space  $T_{[z]} Q^m$  and

$$T_{[z]} Q^m = V(A_{\bar{z}}) \oplus JV(A_{\bar{z}}),$$

where  $V(A_{\bar{z}}) = \mathbb{R}^{m+2} \cap T_{[z]} Q^m$  is the  $(+1)$ -eigenspace and  $JV(A_{\bar{z}}) = i\mathbb{R}^{m+2} \cap T_{[z]} Q^m$  is the  $(-1)$ -eigenspace of  $A_{\bar{z}}$ . That is,  $A_{\bar{z}} X = X$  and  $A_{\bar{z}} JX = -JX$ , respectively, for any  $X \in V(A_{\bar{z}})$ .

Geometrically this means that the shape operator  $A_{\bar{z}}$  defines a real structure on the complex vector space  $T_{[z]} Q^m$ , or equivalently, is a complex conjugation on  $T_{[z]} Q^m$ . Since the real codimension of  $Q^m$  in  $\mathbb{C}P^{m+1}$  is 2, this induces an  $S^1$ -subbundle  $\mathfrak{A}$  of the endomorphism bundle  $\text{End}(TQ^m)$  consisting of complex conjugations.

There is a geometric interpretation of these conjugations. The complex quadric  $Q^m$  can be viewed as the complexification of the  $m$ -dimensional sphere  $S^m$ . Through each point  $[z] \in Q^m$  there exists a one-parameter family of real forms of  $Q^m$  which are isometric to the sphere  $S^m$ . These real forms are congruent to each other under action of the center  $SO_2$  of the isotropy subgroup of  $SO_{m+2}$  at  $[z]$ . The isometric reflection of  $Q^m$  in such a real form  $S^m$  is an isometry, and the differential at  $[z]$  of such a reflection is a conjugation on  $T_{[z]} Q^m$ . In this way the family  $\mathfrak{A}$  of conjugations on  $T_{[z]} Q^m$  corresponds to the family of real forms  $S^m$  of  $Q^m$  containing  $[z]$ , and the subspaces  $V(A) \subset T_{[z]} Q^m$  correspond to the tangent spaces  $T_{[z]} S^m$  of the real forms  $S^m$  of  $Q^m$ .

The Gauss equation for  $Q^m \subset \mathbb{C}P^{m+1}$  implies that the Riemannian curvature tensor  $\bar{R}$  of  $Q^m$  can be described in terms of the complex structure  $J$  and the complex conjugations  $A \in \mathfrak{A}$ :

$$\begin{aligned} \bar{R}(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ \\ &\quad + g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY. \end{aligned}$$

Note that  $J$  and each complex conjugation  $A$  anti-commute, that is,  $AJ = -JA$  for each  $A \in \mathfrak{A}$ .

For every unit tangent vector  $W \in T_{[z]} Q^m$  there exist a conjugation  $A \in \mathfrak{A}$  and orthonormal vectors  $X, Y \in V(A)$  such that

$$W = \cos(t)X + \sin(t)JY$$

for some  $t \in [0, \pi/4]$ . The singular tangent vectors correspond to the values  $t = 0$  and  $t = \pi/4$ . When  $W = X$  for  $X \in V(A)$ ,  $t = 0$ , there exist many kinds of maximal 2-flats  $\mathbb{R}X + \mathbb{R}Z$  for  $Z \in V(A)$  orthogonal to  $X \in V(A)$ . So the tangent vector  $X$  is said to be singular. When  $W = (X + JY)/\sqrt{2}$  for  $t = \pi/4$ , it becomes also a singular tangent vector, which belongs to many kinds of maximal 2-flats given by  $\mathbb{R}(X + JY) + \mathbb{R}Z$  for any  $Z \in V(A)$  orthogonal to  $X \in V(A)$  or  $\mathbb{R}(X + JY) + \mathbb{R}JZ$  for any  $JZ \in JV(A)$ . If  $0 < t < \pi/4$  then the unique maximal flat containing  $W$  is  $\mathbb{R}X \oplus \mathbb{R}JY$ .

### 3. Some general equations

Let  $M$  be a real hypersurface in  $Q^m$  and denote by  $(\phi, \xi, \eta, g)$  the induced almost contact metric structure. Note that  $\xi = -JN$ , where  $N$  is a (local) unit normal vector field of  $M$  and  $\eta$  the corresponding 1-form defined by  $\eta(X) = g(\xi, X)$  for any tangent vector field  $X$  on  $M$ . The tangent bundle  $TM$  of  $M$  splits orthogonally into  $TM = \mathcal{C} \oplus \mathbb{R}\xi$ , where  $\mathcal{C} = \ker(\eta)$  is the maximal complex subbundle of  $TM$ . The structure tensor field  $\phi$  restricted to  $\mathcal{C}$  coincides with the complex structure  $J$  restricted to  $\mathcal{C}$ , and  $\phi\xi = 0$ .

At each point  $z \in M$  we define a maximal  $\mathfrak{A}$ -invariant subspace of  $T_z M, z \in M$  as follows:

$$\mathcal{Q}_z = \{X \in T_z M \mid AX \in T_z M \text{ for all } A \in \mathfrak{A}_z\}.$$

Then we want to introduce an important lemma which will be used in the proof of our main Theorem in the introduction.

**Lemma 3.1** ([19]). *For each  $z \in M$  we have*

- (i) *If  $N_z$  is  $\mathfrak{A}$ -principal, then  $\mathcal{Q}_z = \mathcal{C}_z$ .*
- (ii) *If  $N_z$  is not  $\mathfrak{A}$ -principal, there exist a conjugation  $A \in \mathfrak{A}$  and orthonormal vectors  $X, Y \in V(A)$  such that  $N_z = \cos(t)X + \sin(t)JY$  for some  $t \in (0, \pi/4]$ . Then we have  $\mathcal{Q}_z = \mathcal{C}_z \ominus \mathbb{C}(JX + Y)$ .*

We now assume that  $M$  is a Hopf hypersurface. Then the Reeb vector field  $\xi = -JN$  satisfies the following

$$S\xi = \alpha\xi,$$

where  $S$  denotes the shape operator of the real hypersurfaces  $M$  with the smooth function  $\alpha = g(S\xi, \xi)$  on  $M$ . When we consider the transform  $JX$  by the Kähler structure  $J$  on  $Q^m$  for any vector field  $X$  on  $M$  in  $Q^m$ , we may put

$$JX = \phi X + \eta(X)N$$

for a unit normal  $N$  to  $M$ . Then we now consider the equation of Codazzi

$$g((\nabla_X S)Y - (\nabla_Y S)X, Z) = \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y) \\ + g(X, AN)g(AY, Z) - g(Y, AN)g(AX, Z) + g(X, A\xi)g(JAY, Z) - g(Y, A\xi)g(JAX, Z). \quad (3.1)$$

Putting  $Z = \xi$  in (3.1) we get

$$g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = -2g(\phi X, Y) + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi).$$

On the other hand, we have

$$g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = g((\nabla_X S)\xi, Y) - g((\nabla_Y S)\xi, X) \\ = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((S\phi + \phi S)X, Y) - 2g(S\phi SX, Y).$$

Comparing the previous two equations and putting  $X = \xi$  yields

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi).$$

Reinserting this into the previous equation yields

$$g((\nabla_X S)Y - (\nabla_Y S)X, \xi) = -2g(\xi, AN)g(X, A\xi)\eta(Y) + 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ + 2g(\xi, AN)g(Y, A\xi)\eta(X) - 2g(Y, AN)g(\xi, A\xi)\eta(X) \\ + \alpha g((\phi S + S\phi)X, Y) - 2g(S\phi SX, Y).$$

Altogether this implies

$$0 = 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ + 2g(\xi, AN)g(X, A\xi)\eta(Y) - 2g(X, AN)g(\xi, A\xi)\eta(Y) \\ - 2g(\xi, AN)g(Y, A\xi)\eta(X) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \quad (3.2)$$

At each point  $z \in M$  we can choose  $A \in \mathfrak{A}_z$  such that

$$N = \cos(t)Z_1 + \sin(t)JZ_2$$

for some orthonormal vectors  $Z_1, Z_2 \in V(A)$  and  $0 \leq t \leq \frac{\pi}{4}$  (see Proposition 3 in [9]). Note that  $t$  is a function on  $M$ . First of all, since  $\xi = -JN$ , we have

$$AN = \cos(t)Z_1 - \sin(t)JZ_2, \\ \xi = \sin(t)Z_2 - \cos(t)JZ_1, \\ A\xi = \sin(t)Z_2 + \cos(t)JZ_1. \quad (3.3)$$

This implies  $g(\xi, AN) = 0$  and hence

$$\begin{aligned} 0 &= 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) \\ &\quad + g(X, AN)g(Y, A\xi) - g(Y, AN)g(X, A\xi) \\ &\quad - g(X, A\xi)g(JY, A\xi) + g(Y, A\xi)g(JX, A\xi) \\ &\quad - 2g(X, AN)g(\xi, A\xi)\eta(Y) + 2g(Y, AN)g(\xi, A\xi)\eta(X). \end{aligned} \quad (3.4)$$

#### 4. Reeb invariance and a key lemma

By the equation of Gauss, the curvature tensor  $R(X, Y)Z$  for a real hypersurface  $M$  in  $Q^m$  induced from the curvature tensor  $\bar{R}$  of  $Q^m$  can be described in terms of the complex structure  $J$  and the complex conjugation  $A \in \mathfrak{A}$  as follows:

$$\begin{aligned} R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ &\quad + g(AY, Z)AX - g(AX, Z)AY + g(JAY, Z)JAX - g(JAX, Z)JAY + g(SY, Z)SX - g(SX, Z)SY \end{aligned}$$

for any  $X, Y, Z \in T_z M, z \in M$ .

Now let us put

$$AX = BX + \rho(X)N,$$

for any vector field  $X \in T_z Q^m, z \in M, \rho(X) = g(AX, N)$ , where  $BX$  and  $\rho(X)N$  respectively denote the tangential and the normal component of the vector field  $AX$ . Then  $A\xi = B\xi + \rho(\xi)N$  and  $\rho(\xi) = g(A\xi, N) = 0$ . Then it follows that

$$\begin{aligned} AN &= AJ\xi = JA\xi = -J(B\xi + \rho(\xi)N) \\ &= -(\phi B\xi + \eta(B\xi)N). \end{aligned}$$

The equation gives  $g(AN, N) = -\eta(B\xi)$  and  $g(AN, \xi) = 0$ . From this, together with the definition of the Ricci tensor, we have

$$\text{Ric}(X) = (2m - 1)X - 3\eta(X)\xi - g(AN, N)AX + g(AX, N)AN + g(AX, \xi)A\xi + (\text{Tr}S)SX - S^2X. \quad (4.1)$$

On the other hand, it is known that the Ricci tensor is Reeb invariant, that is,  $\mathcal{L}_\xi S = 0$  if and only if

$$(\phi S - S\phi) \cdot \text{Ric} = \text{Ric} \cdot (\phi S - S\phi). \quad (4.2)$$

Here we want to give a remark as follows:

**Remark 4.1.** Let  $M$  be a real hypersurface over a totally geodesic  $\mathbb{C}P^k \subset Q^{2k}, m = 2k$ . Then by a theorem due to Suh [17] the structure tensor commutes with the shape operator, that is,  $S\phi = \phi S$ . Moreover, the unit normal vector field  $N$  becomes  $\mathfrak{A}$ -isotropic. This gives  $\eta(B\xi) = g(A\xi, \xi) = 0$ . So it naturally satisfies the formula (4.2), that is, Ricci commuting.

On the other hand, from (4.3) we assert an important lemma as follows:

**Lemma 4.2.** Let  $M$  be a Hopf real hypersurface in  $Q^m, m \geq 3$ , with Reeb invariant Ricci tensor. Then the unit normal vector field  $N$  becomes singular, that is,  $N$  is  $\mathfrak{A}$ -isotropic or  $\mathfrak{A}$ -principal.

**Proof.** By putting  $X = \xi$  in (4.2) we get

$$(\phi S - S\phi)\text{Ric}(\xi) = 0. \quad (4.3)$$

Here from (4.1) the Ricci curvature is given by

$$\text{Ric}(\xi) = (2m - 4)\xi - g(AN, N)A\xi + g(A\xi, \xi)A\xi + (\text{tr}S)\alpha\xi - \alpha^2\xi,$$

where  $g(A\xi, \xi) = g(AJN, JN) = -g(JAN, JN) = -g(AN, N)$ . Substituting this one into (4.3) gives

$$g(AN, N)(\phi S - S\phi)A\xi = 0. \quad (4.4)$$

The first case gives that  $g(AN, N) = g(A\xi, \xi) = \cos 2t = 0$ , that is,  $t = \frac{\pi}{4}$ . This implies that the unit normal  $N$  becomes  $N = \frac{X+JY}{\sqrt{2}}$ , which means that  $N$  is  $\mathfrak{A}$ -isotropic.

The second case gives that

$$\phi SA\xi = S\phi A\xi. \quad (4.5)$$

Similarly, we also know that

$$\phi S(AN)^T = S\phi(AN)^T, \quad (4.6)$$

where  $(AN)^T$  denotes the tangential component of the vector field  $AN$  in  $Q^m$ . From these two Eq. (4.5) and we know that the shape operator  $S$  commutes with the structure tensor  $\phi$  on the distribution  $Q^\perp = \text{Span}[A\xi, (AN)^T]$ .



On the other hand, by taking the inner product of (4.5) with the tangent vector field  $A\xi$  we know that

$$S\phi A\xi = \phi SA\xi = 0. \quad (4.7)$$

This gives that

$$SA\xi = \alpha\eta(A\xi)\xi. \quad (4.8)$$

By virtue of the commuting  $S\phi = \phi S$  on the distribution  $Q^\perp = [A\xi, (AN)^T]$ , we know that  $\lambda = 0$  or  $\lambda = \alpha$  if we put  $SA\xi = \lambda AN$ . Moreover, in a paper due to Suh [17] we have mentioned that the distribution  $Q^\perp$  is invariant under the shape operator  $S$  if and only if  $\phi S = S\phi$  on the distribution  $Q^\perp$ . Then, together with the notion of Hopf, without loss of generality we may put

$$S\xi = \alpha\xi, \quad SA\xi = \alpha A\xi, \quad SAN = \alpha AN.$$

From this, together with (4.8), we have for a non-vanishing Reeb function  $\alpha \neq 0$

$$A\xi = \eta(A\xi)\xi = \pm\xi.$$

When the Reeb function  $\alpha$  is vanishing, by the formula in Section 3, that is,

$$Y\alpha = (\xi\alpha)\eta(Y) - 2g(\xi, AN)g(Y, A\xi) + 2g(Y, AN)g(\xi, A\xi),$$

it follows that

$$g(Y, (AN)^T)g(\xi, A\xi) = 0.$$

Since in the second case we have assumed that  $N$  is not  $\mathfrak{A}$ -isotropic, we know  $g(\xi, A\xi) \neq 0$ . So it follows that  $(AN)^T = 0$ . This means that

$$AN = (AN)^T + g(AN, N)N = g(AN, N)N.$$

Then it implies that

$$N = A^2N = g(AN, N)AN = g^2(AN, N)N.$$

This gives that  $g(AN, N) = \pm 1$ , that is, we can take the unit normal  $N$  such that  $AN = N$ . So the unit normal  $N$  is  $\mathfrak{A}$ -principal, that is,  $AN = N$ .  $\square$

In order to prove our main theorem in the introduction, by virtue of Lemma 4.2, we can divide into two classes of hypersurfaces in  $Q^m$  with the unit normal  $N$  is  $\mathfrak{A}$ -principal or  $\mathfrak{A}$ -isotropic. When  $M$  is with  $\mathfrak{A}$ -isotropic, in Section 5 we will give its proof in detail and in Section 6 we will give the remainder proof for the case that  $M$  has a  $\mathfrak{A}$ -principal normal vector field.

## 5. Proof of main theorem with $\mathfrak{A}$ -isotropic

In this section we want to prove our Main Theorem for real hypersurfaces  $M$  in  $Q^m$  with commuting Ricci tensor when the unit normal vector field becomes  $\mathfrak{A}$ -isotropic.

Since we assumed that the unit normal  $N$  is  $\mathfrak{A}$ -isotropic, by the definition in Section 3 we know that  $t = \frac{\pi}{4}$ . Then by the expression of the  $\mathfrak{A}$ -isotropic unit normal vector field, (3.3) gives  $N = \frac{1}{\sqrt{2}}Z_1 + \frac{1}{\sqrt{2}}JZ_2$ . This implies that  $g(A\xi, \xi) = 0$ . Since the unit normal  $N$  is  $\mathfrak{A}$ -isotropic, we know that  $g(\xi, A\xi) = 0$ . Moreover, by (3.4) and using an anti-commuting property  $AJ = -JA$  between the complex conjugation  $A$  and the Kähler structure  $J$ , we proved the following (see also Lemma 4.2 in [19]).

**Lemma 5.1.** *Let  $M$  be a Hopf hypersurface in  $Q^m$  with (local) unit normal vector field  $N$ . For each point  $z \in M$  we choose  $A \in \mathfrak{A}_z$  such that  $N_z = \cos(t)Z_1 + \sin(t)JZ_2$  holds for some orthonormal vectors  $Z_1, Z_2 \in V(A)$  and  $0 \leq t \leq \frac{\pi}{4}$ . Then*

$$0 = 2g(S\phi SX, Y) - \alpha g((\phi S + S\phi)X, Y) - 2g(\phi X, Y) + 2g(X, AN)g(Y, A\xi) - 2g(Y, AN)g(X, A\xi) + 2g(\xi, A\xi)\{g(Y, AN)\eta(X) - g(X, AN)\eta(Y)\}$$

holds for all vector fields  $X, Y$  on  $M$ .

Then for  $\mathfrak{A}$ -isotropic unit normal the Ricci tensor  $S$  of a real hypersurface  $M$  in the complex quadric  $Q^m$  becomes

$$\text{Ric}(X) = (2m - 1)X - 3\eta(X)\xi + g(AX, N)AN + g(AX, \xi)A\xi + hSX - S^2X.$$

From this, together with the fact that  $A\xi = \phi AN$  and  $\phi A\xi = -AN$ , it follows that

$$\phi \cdot \text{Ric}(X) = (2m - 1)\phi X + g(AX, N)A\xi - g(AX, \xi)AN + h\phi SX - \phi S^2X \quad (5.1)$$

and

$$\text{Ric}(\phi X) = (2m - 1)\phi X - g(X, A\xi)AN + g(X, AN)A\xi + hS\phi X - S^2\phi X, \quad (5.2)$$

where the function  $h$  denotes the trace of the shape operator  $S$  of  $M$  in  $Q^m$ . Then subtracting (5.2) from (5.1) gives the following

$$\phi \cdot \text{Ric}(X) - \text{Ric}(\phi X) = h(\phi S - S\phi)X - (\phi S^2 - S^2\phi)X. \quad (5.3)$$

On the other hand, we know that the Reeb invariant Ricci tensor  $\mathcal{L}_\xi \text{Ric} = 0$  is equivalent to the following

$$(\phi S - S\phi) \cdot \text{Ric} = \text{Ric} \cdot (\phi S - S\phi). \quad (5.4)$$

By using the formula (5.4) and taking the trace to (5.3), we have

$$\begin{aligned} \text{Tr}(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi)^2 &= \sum_{i,j} g(\phi \cdot \text{Ric}(e_i) - \text{Ric} \cdot \phi(e_i), \phi \cdot \text{Ric}(e_j) - \text{Ric} \cdot \phi(e_j)) \\ &= h\text{Tr}(\phi S - S\phi)(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi) + \text{Tr}(\phi S^2 - S^2\phi)(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi) \\ &= -\text{Tr}(\phi S^2 - S^2\phi)(\phi \text{Ric} - \text{Ric}\phi), \end{aligned} \quad (5.5)$$

where in the second equality we have used (5.4) and

$$\begin{aligned} \text{Tr}(\phi S - S\phi)(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi) &= \text{Tr}\phi \cdot \text{Ric}(\phi S - S\phi) - \text{Tr}(\phi S - S\phi)\text{Ric} \cdot \phi \\ &= \text{Tr}\phi(\phi S - S\phi) \cdot \text{Ric} - \text{Tr}(\phi S - S\phi)\text{Ric} \cdot \phi \\ &= \text{Tr}(\phi S - S\phi)\text{Ric} \cdot \phi - \text{Tr}(\phi S - S\phi)\text{Ric} \cdot \phi \\ &= 0. \end{aligned}$$

On the other hand, the final term in (5.5) becomes the following

$$\begin{aligned} \text{Tr}(\phi S^2 - S^2\phi)(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi) &= \text{Tr}\phi S^2\phi \cdot \text{Ric} - \text{Tr}S^2\phi^2 \cdot \text{Ric} - \text{Tr}\phi S^2\text{Ric} \cdot \phi + \text{Tr}S^2\phi \cdot \text{Ric} \cdot \phi \\ &= 2\text{Tr}\phi S^2\phi \cdot \text{Ric} - \text{Tr}S^2\phi^2 \cdot \text{Ric} - \text{Tr}\phi S^2\text{Ric} \cdot \phi. \end{aligned} \quad (5.6)$$

By the property (5.4) due to the Reeb invariant Ricci tensor  $\mathcal{L}_\xi \text{Ric} = 0$ , we have

$$\phi S(\phi S \cdot \text{Ric} - \text{Ric} \cdot \phi S + \text{Ric} \cdot S\phi - S\phi \text{Ric}) = 0.$$

From this, by taking the trace, the first two terms become

$$\text{Tr}(\phi S)^2 \cdot \text{Ric} - \text{Tr}\phi S \cdot \text{Ric} \cdot \phi S = \text{Tr}(\phi S)^2 \text{Ric} - \text{Tr}(\phi S)^2 \text{Ric} = 0.$$

Then taking the trace of the other two terms becomes

$$\text{Tr}\phi S \cdot \text{Ric} \cdot S\phi = \text{Tr}\phi S^2\phi \cdot \text{Ric}.$$

By virtue of this equation and using the notion of Hopf, the formula (5.5) can be changed as follows:

$$\begin{aligned} \text{Tr}(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi)^2 &= -\text{Tr}(\phi S^2 - S^2\phi)(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi) \\ &= \text{Tr}\phi^2 \cdot \text{Ric} \cdot S^2 + \text{Tr}\phi^2 S^2 \cdot \text{Ric} - 2\text{Tr}\phi^2 S \cdot \text{Ric} \cdot S \\ &= 0, \end{aligned} \quad (5.7)$$

where we have used the following equations

$$\begin{aligned} \text{Tr}\phi^2 \cdot \text{Ric} \cdot S^2 &= \text{Tr}(-\text{Ric} \cdot S^2 + \eta(\text{Ric} \cdot S^2)\xi) \\ &= -\text{Tr}\text{Ric} \cdot S^2 + \eta(\text{Ric}(S^2\xi)), \end{aligned} \quad (5.8)$$

$$\begin{aligned} \text{Tr}\phi^2 \cdot S^2 \cdot \text{Ric} &= \text{Tr}(-S^2 \cdot \text{Ric} + \eta(S^2 \cdot \text{Ric})\xi) \\ &= -\text{Tr}\text{Ric} \cdot S^2 + \eta(S^2 \cdot \text{Ric}\xi), \end{aligned} \quad (5.9)$$

and

$$\begin{aligned} -2\text{Tr}\phi^2 S \cdot \text{Ric} \cdot S &= -2\text{Tr}(-S \cdot \text{Ric} \cdot S + \eta(S^2 \cdot \text{Ric})\xi) \\ &= 2\text{Tr}S \cdot \text{Ric} \cdot S - 2\eta(S \cdot \text{Ric}(S\xi)). \end{aligned} \quad (5.10)$$

From this we conclude that the Ricci tensor  $\text{Ric}$  commutes with the structure tensor  $\phi$  for a case where the unit normal  $N$  is  $\mathfrak{A}$ -isotropic. Then by a theorem due to Suh and Hwang [18], we give a complete classification in our main Theorem in the introduction.



## 6. Proof of main theorem with $\mathfrak{A}$ -principal

In this section we want to prove our Main Theorem for real hypersurfaces in the complex quadric  $Q^m$  with commuting Ricci tensor and  $\mathfrak{A}$ -principal unit normal vector field. By the Ricci tensor given in the formula (4.1) for  $\mathfrak{A}$ -principal unit normal, we give the following

$$\text{Ric}(\phi X) = (2m - 1)\phi X - g(AN, N)A\phi X + g(A\phi X, N)AN + hS\phi X - S^2\phi X, \quad (6.1)$$

and

$$\phi \text{Ric}(X) = (2m - 1)\phi X - g(AN, N)\phi AX + g(AX, N)\phi AN + h\phi SX - \phi S^2X, \quad (6.2)$$

where the function  $h$  denotes the trace of the shape operator  $S$  of  $M$  in  $Q^m$ .

When we consider the unit normal  $N$  is  $\mathfrak{A}$ -principal, the unit normal  $N$  is invariant under the complex conjugation  $A$  in  $\mathfrak{A}$ , that is,  $AN = N$  and  $A\xi = -\xi$ . By using such properties into (6.1) and (6.2), we have

$$\phi \cdot \text{Ric}(X) - \text{Ric} \cdot \phi(X) = -\phi AX + A\phi X + h(\phi S - S\phi)X - (\phi S^2 - S^2\phi)X.$$

From this, together with  $\mathcal{L}_\xi \text{Ric} = 0$ , which is equivalent to  $(\phi S - S\phi) \cdot \text{Ric} = \text{Ric} \cdot (\phi S - S\phi)$ , we have

$$\begin{aligned} \text{Tr}(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi)^2 &= h\text{Tr}(\phi S - S\phi)(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi) \\ &\quad - \text{Tr}(\phi S^2 - S^2\phi)(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi) - \text{Tr}(\phi A - A\phi)(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi). \end{aligned}$$

On the other hand, the complex conjugation is involutive and anti-commuting such that  $AJ = -JA$ , and the unit normal  $N$  is  $\mathfrak{A}$ -invariant, it follows that

$$\phi A = -A\phi.$$

From this, together with  $A\xi = -\xi$ , we have

$$\begin{aligned} \text{Tr}\phi A(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi) &= -\text{Tr}A\phi^2 \cdot \text{Ric} - \text{Tr}\text{Ric} \cdot \phi^2 A \\ &= 2\text{Tr}\text{Ric} \cdot A - \eta(\text{Ric}(A\xi)) - \eta(A \cdot \text{Ric}(\xi)) \\ &= 2\{\text{Tr}\text{Ric} \cdot A + \eta(\text{Ric}(\xi))\}. \end{aligned}$$

Then it follows that

$$\begin{aligned} \text{Tr}(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi)^2 &= -\text{Tr}(\phi S^2 - S^2\phi)(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi) - \text{Tr}(\phi A - A\phi)(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi) \\ &= 2\eta(\text{Ric} \cdot S^2(\xi)) - 2\eta(S \cdot \text{Ric} \cdot S(\xi)) - 4\text{Tr}(\text{Ric} \cdot A) - 4\eta(\text{Ric}(\xi)). \end{aligned} \quad (6.3)$$

The Ricci tensor given in the formula (4.1) for  $\mathfrak{A}$ -principal unit normal, that is,  $A\xi = -\xi$  gives the following

$$\text{Ric}(X) = (2m - 1)X - 2\eta(X)\xi - AX + hSX - S^2X,$$

and

$$\text{Ric}(\xi) = \{2(m - 1) + h\alpha - \alpha^2\}\xi.$$

Then it follows that

$$\text{Ric}(e_i) = (2m - 1)e_i - 2\eta(e_i)\xi - Ae_i + hSe_i - S^2e_i,$$

and

$$\text{Ric}(Ae_i) = (2m - 1)e_i + 2\eta(e_i)\xi - e_i + hSAe_i - S^2Ae_i,$$

where we have taken an orthonormal basis  $\{\xi, e_1, \dots, e_{m-1}, \phi e_1, \dots, \phi e_{m-1}\}$  of  $T_z M$ ,  $z \in M$ , in  $Q^m$  such that  $Ae_i = e_i$ ,  $A\phi e_i = -\phi e_i$ ,  $A\xi = -\xi$  and  $AN = N$ . So it follows that

$$\begin{aligned} \text{Tr}(\text{Ric} \cdot A) &= g(A\xi, \text{Ric}(\xi)) + \sum_{i=1}^{2m-2} g(Ae_i, \text{Ric}(e_i)) \\ &= -g(\xi, \text{Ric}(\xi)) + \sum_{i=1}^{m-1} g(Ae_i, \text{Ric}(e_i)) + \sum_{i=1}^{m-1} g(A\phi e_i, \text{Ric}(\phi e_i)). \end{aligned}$$

Substituting these ones into (6.3) and using the orthonormal basis, we have

$$\begin{aligned} \text{Tr}(\phi \cdot \text{Ric} - \text{Ric} \cdot \phi)^2 &= -4 \sum_{i=1}^{m-1} \{g(\text{Ric}(e_i), e_i) - g(\phi e_i, \text{Ric}(\phi e_i))\} \\ &= -4\{\text{Tr}^* \text{Ric} + \text{Tr}^* \phi \cdot \text{Ric} \cdot \phi\} \\ &= -4\{\text{Tr}^* \text{Ric} + \text{Tr}^* \phi^2 \cdot \text{Ric}\} \\ &= -4\{\text{Tr}^* \text{Ric} - \text{Tr}^* \text{Ric}\} \\ &= 0, \end{aligned} \quad (6.4)$$

where  $Tr^*Ric$  denotes  $Tr^*Ric = \sum_{i=1}^{m-1} g(Ric(e_i), e_i)$  for the orthonormal basis  $\{\xi, e_1, \dots, e_{m-1}, \phi e_1, \dots, \phi e_{m-1}\}$  of  $T_z M$ ,  $z \in M$ , in  $Q^m$ . This concludes that even for the  $\mathfrak{A}$ -principal normal the Ricci tensor  $Ric$  commutes with the structure tensor  $\phi$ , that is,  $Ric \cdot \phi = \phi \cdot Ric$ . Then by Theorem C due to Suh and Hwang [18], we give a complete classification of our main result.

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